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Non-integrability of some Hamiltonian systems in polar coordinates

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Abstract. In this paper we first reformulate a non-integrability criterion obtained by Yoshida for Hamiltonian systems with two degrees of freedom in order to make it easier to handle those problems whose natural formulation is given in polar coordinates, as occurs with those that have harmonic potential. Among other applications, we prove the non-integrability of the satellite problem under McCullagh's approximation of the potential, i.e. truncated at the r^{-3} term that, in most cases, is the main problem of the satellite of a triaxial primary body, hence its importance.

1. Introduction

Recently, the non-integrability criteria obtained by Yoshida (1987, 1989), on the basis of Ziglin's result (1983), have been successfully applied to some problems in satellite dynamics. In this way, Irigoyen and Simó (1993) established the non-integrability of the J_2-problem, while Ferrándiz and Sansaturio (1995) proved the non-existence of rational integrals in the J_{22}-problem and, more recently, Ferrándiz *et al* (1996) showed the non-integrability of that of the general zonal satellite truncated at any order.

However, calculations normally turn out to be rather complicated since the potential function of the satellite, as well as that of a wide class of problems in classical field theory, have a natural expansion in spherical harmonics, while Yoshida's criteria are formulated in Cartesian coordinates and require that the kinetic energy T is a quadratic form of the type $T = \frac{1}{2}|\boldsymbol{p}|^2$.

Facing the application to the satellite problem and others of interest in dynamics, it seems convenient to obtain analogous formulations to those of Yoshida but in polar or spherical coordinates, especially when dealing with harmonics of a high order. It is clear that, as the available criterion depends on whether the number of degrees of freedom is 2 or $n > 2$, the treatment of the planar case in polar coordinates or that of the spatial case in spherical ones must be different.

In this paper, we reformulate Yoshida's theorem (1987) in polar coordinates and apply it to some subproblems of that of the satellite with a restricted perturbation. As a first simple example, it is proved that any sectorial perturbation, when considered in isolation, gives

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rise to a non-integrable problem (in the Liouville sense) through first integrals for which quite little demanding regularity conditions are required.

The second example is much more interesting and, probably, expected. It concerns the non-integrability of the problem of a satellite orbiting around a rigid body of arbitrary shape, when the potential is truncated at the term containing the factor r^{-3}, i.e. what is sometimes referred to as McCullagh's potential. This approximation of the potential can be envisaged as generating the general main problem of the satellite of a triaxial primary. It is well known that, unlike what happens in the Earth, for many celestial bodies (the moon being the most emblematic in the solar system) the second zonal harmonic does not give a good representation of the potential and thus, the first approach should include all the harmonics of order 2.

Finally, we also include a third example in which we show the convenience of using our criterion to establish the non-integrable cases of a generalized Yang–Mills Hamiltonian.

The non-integrability criterion in polar coordinates reads as follows.

Theorem 1. Let $U(r,\theta) = r^k\, W(\theta)$ be a potential function, k being an integer but $k \neq \pm 2,\ 0$, and compute the quantity λ defined by

$$\lambda = \frac{W_{\theta\theta}(\theta_0)}{k W(\theta_0)}$$

where θ_0 is a root of the equation $W_\theta(\theta) = 0$ such that $W(\theta_0) \neq 0$ and the variable subscripts denote partial derivatives.

If λ lies in the regions S_k defined below, then the Hamiltonian system

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + U\,(r,\theta)$$

is non-integrable, i.e. there cannot exist an additional integral Φ which is complex analytic in $(r, \theta, p_r, p_\theta)$. The regions S_k are defined as follows

(i) $k \geqslant 3$

$$S_k = (-\infty, -1) \cup \left(\bigcup_{m\in\mathbb{N}} \left(\frac{(m-1)(mk+2)}{2}, \frac{(m+1)(mk-2)}{2}\right)\right) \tag{1}$$

(ii) $k = 1$

$$S_1 = \mathbb{R} - \left\{-1, 0, 2, 5, 9, \ldots, \frac{m(m+1)}{2} - 1, \ldots; m \in \mathbb{N}\right\} \tag{2}$$

(iii) $k = -1$

$$S_{-1} = \mathbb{R} - \left\{0, -1, -3, -6, -10, \ldots, -\frac{m(m+1)}{2}, \ldots; m \in \mathbb{N}\right\} \tag{3}$$

(iv) $k \leqslant -3$

$$S_k = \mathbb{R}^+ \cup \left(\bigcup_{m\in\mathbb{N}} \left(-\frac{m(m-1)|k|}{2} - m, -\frac{m(m+1)|k|}{2} + m\right)\right). \tag{4}$$

2. Yoshida's criterion in polar coordinates

Yoshida considered the canonical system

$$\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} = \frac{\partial H}{\partial \boldsymbol{p}} \qquad \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = -\frac{\partial H}{\partial \boldsymbol{q}} \tag{5}$$

with Hamiltonian

$$H = \tfrac{1}{2}\boldsymbol{p}^2 + U(\boldsymbol{q}) \tag{6}$$

where $\boldsymbol{p} = (p_1, p_2)$, $\boldsymbol{q} = (q_1, q_2)$ and the potential function $U(\boldsymbol{q})$ is homogeneous of an integer degree k. By this assumption, equations (5) always have a straight-line solution of the form

$$\boldsymbol{q} = \boldsymbol{c}\varphi(t) \qquad \boldsymbol{p} = \boldsymbol{c}\dot{\varphi}(t)$$

where $\varphi(t)$ is a solution of the nonlinear differential equation $\ddot{\varphi}(t) + \varphi^{k-1} = 0$ and the constant vector $\boldsymbol{c} = (c_1, c_2)$ is a solution of the algebraic equation $\boldsymbol{c} = U_{\boldsymbol{q}}(\boldsymbol{c})$ in the complex field.

Under these hypotheses, the non-integrability criterion obtained by Yoshida claims that: 'If the quantity $\lambda_Y = \mathrm{Tr}\, U_{\boldsymbol{qq}}(c_1, c_2) - (k-1)$, where $U_{\boldsymbol{qq}}$ is the Hessian matrix of $U(q_1, q_2)$ and $k \neq \pm 2, 0$, lies in the so-called non-integrability regions (Yoshida, 1987; p 128, equation (1.3)), then the 2-degrees-of-freedom Hamiltonian (6) is non-integrable.' The quantity λ_Y is referred to as the integrability coefficient.

Now, by performing the canonical transformation to polar coordinates $(r, \theta, p_r, p_\theta)$, the Hamiltonian (6) becomes

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + U(r, \theta). \tag{7}$$

If we assume the potential function $U(r, \theta)$ to be of the type $U(r, \theta) = r^k\, W(\theta)$, where k is an integer but $k \neq \pm 2, 0$, the corresponding equations of motion are

$$\frac{\mathrm{d}r}{\mathrm{d}t} = p_r \qquad \frac{\mathrm{d}p_r}{\mathrm{d}t} = \frac{p_\theta^2}{r^3} - k r^{k-1} W$$
$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{p_\theta}{r^2} \qquad \frac{\mathrm{d}p_\theta}{\mathrm{d}t} = -r^k W_\theta.$$

This system admits a solution of the form

$$\begin{aligned} \theta &= \theta_0 \qquad & r &= c\varphi(t) \\ p_\theta &= 0 \qquad & p_r &= c\dot{\varphi}(t) \end{aligned} \tag{8}$$

where θ_0 is a root of $W_\theta(\theta) = 0$ such that $W(\theta_0) \neq 0$, $\varphi(t)$ is a solution of the nonlinear differential equation $\ddot{\varphi} + \varphi^{k-1} = 0$ with initial conditions $\varphi(0) = 1, \dot{\varphi}(0) = 0$ and the constant c is a solution of the algebraic equation

$$1 = k\, c^{k-2} W(\theta_0). \tag{9}$$

From the equations of the transformation

$$q_1 = r\cos\theta$$
$$q_2 = r\sin\theta$$

we easily obtain

$$\begin{aligned} \frac{\partial^2 U}{\partial q_1^2} + \frac{\partial^2 U}{\partial q_2^2} &= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right)\right] + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} \\ &= k^2 r^{k-2} W(\theta) + r^{k-2} W_{\theta\theta}(\theta). \end{aligned}$$

On the other hand

$$U_{\boldsymbol{qq}}(\boldsymbol{c}) = U_{\boldsymbol{qq}}(\boldsymbol{q})\big|_{t=0} = U_{\boldsymbol{qq}}(\boldsymbol{q}_\mathrm{o}).$$

In our case,

$$\boldsymbol{q}_{\mathbf{o}} = (r_0 \cos\theta_0, r_0 \sin\theta_0) \qquad \text{with } r_0 = c\varphi(0) = c.$$

Hence,

$$\begin{aligned}\lambda_Y &= \operatorname{Tr} U_{\boldsymbol{qq}}(\boldsymbol{c}) - (k-1) = U_{q_1q_1}(c,\theta_0) + U_{q_2q_2}(c,\theta_0) - (k-1)\\ &= k^2 c^{k-2} W(\theta_0) + c^{k-2} W_{\theta\theta}(\theta_0) - (k-1).\end{aligned}$$

But according to (9)

$$1 = W(\theta_0) k c^{k-2} \qquad k = k^2 W(\theta_0) c^{k-2}$$

and therefore,

$$\lambda_Y = 1 + c^{k-2} W_{\theta\theta}(\theta_0) = 1 + \frac{W_{\theta\theta}(\theta_0)}{k W(\theta_0)}.$$

The quantity

$$\lambda = \frac{W_{\theta\theta}(\theta_0)}{k W(\theta_0)}$$

can be considered as the integrability coefficient in polar coordinates and it relates to Yoshida's integrability coefficient through the equation $\lambda = \lambda_Y - 1$, so that the non-integrability regions S_k obtained by Yoshida transform accordingly and give rise to the non-integrability regions defined by equations (1)–(4).

3. The sectorial harmonic potential

Let us consider the satellite problem with the restriction that the only acting perturbation is that due to an arbitrary sectorial term.

If we assume that the satellite is in equatorial orbit, the Hamiltonian of the problem can be written as

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + r^{-(n+1)}\, W(\theta) \qquad n \geqslant 2 \tag{10}$$

where $W(\theta) = \cos n\theta$ with a suitable choice of the axes. In this case, the equation $W_\theta(\theta) = 0$ admits $\theta_0 = 0$ as solution.

Therefore, the integrability coefficient λ happens to be

$$\lambda = \frac{W_{\theta\theta}(\theta_0)}{k W(\theta_0)} = \frac{-n^2 \cos n\theta_0}{-(n+1)\cos n\theta_0} = \frac{n^2}{n+1} > 0.$$

According to our theorem 1, as $\mathbb{R}^+ \subset S_k$ for $k \leqslant -3$, it follows that $\lambda \in S_k$ and hence, the Hamiltonian (10) is non-integrable in the Liouville sense, i.e. there cannot exist an additional meromorphic integral independent of the Hamiltonian itself.

Let us remark that the theory presented here would allow us to establish the non-integrability of the planar J_{22}-problem in a much simpler way than that carried out by Ferrándiz and Sansaturio (1995). The reader is really encouraged to compare both procedures in order to appreciate the convenience of this new formulation.

4. The $(J_2 + J_{22})$-problem or 'general main satellite problem'

In this example we will address the problem of the motion of a particle attracted by a rigid body of an arbitrary shape under the assumption that all the terms beyond r^{-3} in the potential expansion in spherical harmonics are neglected. In addition, we will also assume that the central body rotates with constant angular velocity ω.

The resulting Hamiltonian can be considered as the 'main problem' for the satellites of many celestial bodies for which the differences, $C - A$, $B - A$, between the moments of inertia are of the same order of magnitude.

It is immediate that the problem admits planar solutions, corresponding to the equatorial orbits, so that the Hamiltonian can be restricted to this 2-degrees-of-freedom case. Using polar coordinates, in the extended phase space such a Hamiltonian is given by

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \frac{\mu}{r} - \omega p_\theta + V_2 + V_{22} + p_0 = 0 \tag{11}$$

where

$$V_2 = \frac{\varepsilon_2}{r^3}\,[P_2(z)]_{z=0} = \frac{\varepsilon_2}{r^3}\left[\frac{1}{2}\left(3z^2 - 1\right)\right]_{z=0} = -\frac{\varepsilon_2}{2r^3} \tag{12}$$

$$V_{22} = \frac{\varepsilon_{22}}{r^3}\cos\left(2\theta - 2\theta_{22}\right) \tag{13}$$

and we have adopted the usual notations for Earth satellites, so that the small parameters in (12) and (13) are given by $\varepsilon_2 = \mu J_2 R^2$, $\varepsilon_{22} = \mu J_{22} R^2$, in terms of the adimensional coefficients J_2, J_{22} and the constants μ (the reduced mass) and R (a reference radius of the body).

Note that, when $\omega = 0$ and with a different meaning of the constants, the Hamiltonian (11) can also describe the classical motion of a particle in the electrostatic field generated by a uniformly charged ellipsoid.

We can assume that $\theta_{22} = 0$, which is equivalent to a suitable choice of the axes in the inertial OXY plane and allows us to simplify $V_{22} = \varepsilon_{22} r^{-3}\cos 2\theta$. Note that J_2 and J_{22} are both greater than zero for the Earth (or for any body with $A < B < C$), and so are ε_2 and ε_{22}. In the general case, we can assume that ε_2 and ε_{22} have the same sign, since the sign of ε_{22} can be reversed by means of a $\pi/2$ rotation of the axes.

Now we cannot apply the non-integrability criterion directly since the Hamiltonian (11) is not of the required type. Moreover, the procedure in section 2 cannot be followed because a solution of the form (8) is not available. In order to overcome this drawback we will reduce our problem to an auxiliary one with the suitable form.

By performing the change of scale $(t, r, \theta; p_0, p_r, p_\theta) \longrightarrow (\bar{t}, \bar{r}, \bar{\theta}; \bar{p}_0, \bar{p}_r, \bar{p}_\theta)$ defined by

$$\begin{aligned} t &= \beta\bar{t} \qquad & p_0 &= \beta^{-\frac{6}{5}}\bar{p}_0 \\ r &= \beta^{\frac{2}{5}}\bar{r} \qquad & p_r &= \beta^{-\frac{3}{5}}\bar{p}_r \\ \theta &= \bar{\theta} \qquad & p_\theta &= \beta^{-\frac{1}{5}}\bar{p}_\theta \end{aligned} \tag{14}$$

and removing the bar in the new variables for the sake of simplicity in the notation, after straightforward calculations, as $\beta \to 0$, (11) becomes

$$K = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + r^{-3}W(\theta) + p_0 = 0 \tag{15}$$

with

$$W(\theta) = -\frac{\varepsilon_2}{2} - \varepsilon_{22}\cos 2\theta$$

which is taken as the Hamiltonian of an auxiliary problem.

In this case, $W_\theta(\theta) = 2\varepsilon_{22}\sin 2\theta$ and $W_{\theta\theta}(\theta) = 4\varepsilon_{22}\cos 2\theta$, therefore the equation $W_\theta(\theta) = 0$ admits $\theta_0 = 0$ as solution and the integrability coefficient λ turns out to be

$$\lambda = \frac{W_{\theta\theta}(\theta_0)}{kW(\theta_0)} = \frac{4\varepsilon_{22}}{3(\varepsilon_2/2 + \varepsilon_{22})} > 0.$$

According to theorem 1, as $\mathbb{R}^+ \subset S_k$ for $k \leqslant -3$, it follows that $\lambda \in S_k$ and hence, the Hamiltonian (15) is non-integrable in the Liouville sense, which proves the non-existence of any additional meromorphic integral. In the case of $\varepsilon_{22} = 0$, the integrability coefficient vanishes, but it corresponds to the main problem of the zonal satellite (the J_2-problem) studied by Irigoyen and Simó (1993).

Note that the set of variables used is not relevant whenever the transformation from polar variables to new ones is meromorphic.

From the result obtained for the auxiliary problem, it is not difficult to conclude the non-existence of additional rational integrals in the original one (11). Let us first note that if $\Phi(\boldsymbol{p}, \boldsymbol{q}, t) =$ constant is a rational first integral, by performing the change of variables (14), it becomes

$$\overline{\Phi}(\overline{\boldsymbol{p}}, \overline{\boldsymbol{q}}, \overline{t}) = \frac{\sum_m \beta^m f_m}{\sum_n \beta^n f_n} = \text{constant} \tag{16}$$

where f_m and f_n are polynomials in $\overline{\boldsymbol{p}}$, $\overline{\boldsymbol{q}}$, t.

Expression (16) is a rational first integral of (11). Then, multiplying by an adequate power of β and taking limits as $\beta \to 0$, we obtain a rational integral of the auxiliary problem (15).

In the light of the fact that (15) does not have any additional meromorphic integral, it cannot admit any additional rational integral and thus, neither does (11).

As for the spatial problem, it cannot be completely integrable through rational integrals if we require them not to be singular on the plane $z = 0$.

As a summary we can state the following conclusion.

The ($J_2 + J_{22}$)-problem does not have any additional global first integral that is rational.

Note that transformation (14) would also provide the same reduced Hamiltonian (15) if the original problem is that of the classical motion of a particle in a field created by an electrostatic quadrupole (suitably oriented) and a uniform magnetic field $\boldsymbol{B}$ in the z-direction. Hence, the previous assertions are also meaningful for this problem.

Remark. Following the same procedure as in the previous example it is also possible to establish the rational non-integrability of any Hamiltonian of the form

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \frac{\mu}{r} + \sum_{i=-n}^{M} \frac{1}{r^i} W_i(\theta) \tag{17}$$

n being any integer such that $n \leqslant M$ and $M > 2$, provided the problem described by the Hamiltonian

$$H_M = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \frac{\mu}{r} + \frac{1}{r^M} W_M(\theta)$$

turns out to be non-integrable.

This could be proved by performing a suitable change of scale, similar to (14), so that the non-existence of rational integrals is obtained by following the same steps as before. Likewise, it would also be possible to establish the non-existence of meromorphic integrals by means of Yoshida's theorem for potential functions that are sums of homogeneous terms (Yoshida, 1988; theorem 4.1), provided the Hamiltonian (17) has a straight-line solution.

Let us finally point out that this would be the case for any satellite problem for which the perturbed potential consists of a finite sum of even zonal terms and a finite sum of sectorial ones. An example involving only terms of the first type was obtained by Irigoyen (1996), who directly proved the non-existence of integrals for any truncation of the two fixed centres problem in the symmetric case. However, when the last term is an odd zonal harmonic, the procedure should be modified according to the authors' results (Ferrándiz *et al* 1996).

5. A generalized Yang–Mills Hamiltonian

In this example we pay attention to a Yang–Mills-type Hamiltonian

$$H = \tfrac{1}{2}\left(p_1^2 + p_2^2 + a_1 q_1^2 + a_2 q_2^2\right) + \tfrac{1}{4} q_1^4 + \tfrac{1}{4} a_3 q_2^4 + \tfrac{1}{2} a_4 q_1^2 q_2^2. \tag{18}$$

This Hamiltonian also appears in connection with some problems in scalar field theory (cf Fridberg *et al* 1976) and in the semiclassical method in quantum field theory (cf Rajaraman and Weinberg 1975). Its integrability has been studied by several authors (Bountis *et al* 1982, Ziglin 1983, Yoshida 1986, Villarroel 1988, Ichtiaroglou 1989) and, more recently, Kasperczuk (1994) and Elipe *et al* (1995) presented the points $a \in \mathbb{R}^4$ providing the five integrable cases of this problem which are known so far. They correspond to the values

(A) $a_1 = a_2, a_3 = a_4 = 1$,
(B) $a_1 = a_2, a_3 = 1, a_4 = 3$,
(C) $a_2 = 4a_1, a_3 = 16, a_4 = 6$,
(D) $a_4 = 0$,
(E) $a_2 = 4a_1, a_3 = 8, a_4 = 3$.

The Hamiltonian (18) can be easily handled in polar coordinates to study those cases which are definitively non-integrable. Following the same procedure as in the previous section, the change of scale

$$\begin{aligned} &t = \beta \bar{t} \\ &q_1 = \beta^{-1}\bar{q}_1 \qquad p_1 = \beta^{-2}\bar{p}_1 \\ &q_2 = \beta^{-1}\bar{q}_2 \qquad p_2 = \beta^{-2}\bar{p}_2 \end{aligned}$$

provides, as $\beta \to 0$, an auxiliary Hamiltonian with quartic potential

$$K = \tfrac{1}{2}(p_1^2 + p_2^2) + \tfrac{1}{4} q_1^4 + \tfrac{1}{4} a_3 q_2^4 + \tfrac{1}{2} a_4 q_1^2 q_2^2 \tag{19}$$

which, after performing the change to polar coordinates, turns out to be

$$K = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + r^4 W(\theta) \tag{20}$$

where

$$W(\theta) = \tfrac{1}{4}\cos^4\theta + \tfrac{1}{4} a_3 \sin^4\theta + \tfrac{1}{2} a_4 \sin^2\theta \cos^2\theta.$$

Straightforward calculations provide $\theta_0 = 0$ as a solution to $W_\theta(\theta_0) = 0$ and the integrability coefficient is

$$\lambda = \frac{W_{\theta\theta}(\theta_0)}{4W(\theta_0)} = a_4 - 1.$$

Therefore, as $k = 4$ in this case, according to (1) the non-integrability regions are

$$(-\infty, -1) \cup (0, 2) \cup (5, 9) \cup (14, 20) \cup \ldots.$$

That is, whenever a_4 belongs to

$$(-\infty, 0) \cup (1, 3) \cup (6, 10) \cup (15, 21) \cup \ldots$$

and a_3 remains arbitrary, the Hamiltonian (20) does not have any additional meromorphic first integral.

By means of similar arguments to those used in the previous example, the non-existence of any additional meromorphic integral for the problem described by (19) (or (20)) leads to the non-existence of any additional first integral of (18) that is rational.

We draw the reader's attention to the fact that the aforementioned five known integrable cases are placed just in some extremes of the non-integrability intervals. It seems interesting to investigate this further, although the analysis could be rather involved. Note that the Hamiltonian (20) has another straight-line solution at $\theta_0 = \pi/2$, providing the integrability coefficient $\lambda_2 = a_4/a_3 - 1$. Despite the obvious overlapping for $a_3 = 1$, there exists no inclusion relation between both cases. For instance, for $a_4 = 2$ and $a_3 = \frac{1}{2}$ the solution at $\theta_0 = 0$ ensures the non-integrability, while $\theta_0 = \pi/2$ gives $\lambda_2 = 4$ and it does not provide any information. Nevertheless, when $a_4 = 4$ and $a_3 = 2$, the solution at $\theta_0 = 0$ is not useful for establishing the non-integrability, while for $\theta_0 = \pi/2$ we obtain $\lambda_2 = 1$ and the non-integrability follows. On the other hand, in the previous (C) and (E) cases, the integrability coefficient for the straight-line solution $\theta_0 = \pi/2$ is $\lambda_2 = -\frac{5}{8}$, which is not a limit value of the non-integrability region.

We are currently working on this subject and we expect to report on the results soon.

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